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A Mixture of Fuzzy Filters Applied to the Analysis of Heartbeat Intervals

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Abstract This study provides a stochastic modeling of the heartbeat intervals using a mixture of Takagi-Sugeno type fuzzy filters. The model parameters are inferred under variational Bayes (VB) framework. The model of the heartbeat intervals is in the form of a history-dependent probability density. The parameters, characterizing the heartbeat intervals probability density, include the estimated parameters of different fuzzy filters and may serve as the features of the heartbeat interval series. The features of the heartbeat intervals provide a description of the physiological state of an individual. A novelty of our analysis method is that the physiological state is predicted as a part of the features extraction procedure. This is done via deriving, using VB paradigm, an analytical expression for the posterior distribution that the observed heartbeat intervals have been generated by the stochastic model of the physiological state. The method is illustrated with the data of 40 healthy subjects studied in a tilt-table experiment.

Keywords Fuzzy filtering · R-R interval · Heart rate variability · Variational Bayes · Probability distribution

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1 Introduction

Heart rate variability (HRV), being an important quantitative measure of cardiovascular regulation by the autonomic nervous system, is one of the most frequently studied physiological rhythms. HRV is described with a mathematical analysis of the R-R intervals on the electrocardiogram (ECG). A large number of analysis methods have been developed to characterize HRV. The available methods include statistical measures analysis [39]; spectral analysis [1,32,6]; nonlinear dynamics [40]; point-process modeling [5,4]. The authors in [5,4] give a probabilistic definition of HRV by developing a history-dependent inverse Gaussian point process model of heartbeat intervals. Adaptive filtering algorithms can be used to estimate the parameters of the heartbeat intervals probabilistic model [4].

The mathematical analysis of physiological signals remains as an active area of research [36,11,14,2,9,13,4,43,31,7]. The mathematical analysis results in the extraction of signal features such that the features should characterize the signal under different physiological conditions of the individuals. However, it is typical that the correlations between signal features and the patients' physiological states are complex and uncertain. Thus, the attempts have been made to approximate these complex and uncertain correlations through the neural/fuzzy models [10,12,35,19,18,42,41,21,20,8,29,15,24,23]. Adaptive filters are typically used to remove noise and artifacts from the biomedical signals [33,26,34,30,28]. The adaptive filtering algorithms for linear models have been well studied [37]. However, the studies dealing with the mathematical theory addressing the issues of robustness, convergence, and steady-state error for nonlinear neuro/fuzzy filters are still relatively few available. We have recently done some work to develop fuzzy filtering based methods for the analysis of physiological data [24,16,22].

The study of fuzzy filtering approach, with the goal of providing a model that gives a highly accurate description of the heartbeat intervals at rest and in extreme physiological conditions, is the aim of this work. We feel that an integrated approach that combines fuzzy filtering with the stochastic methods has much to offer in developing the reliable physiological models. Our method of heartbeat intervals (or R-R intervals) analysis is derived from the following ideas:

- Given the history of heartbeat intervals, the current interval is modeled as a Gaussian distribution whose mean and variance are further random variables.
- The mean of the distribution is given by the output of a fuzzy filter. The history of the intervals serve as the inputs to the fuzzy filter. The output of the fuzzy filter is random in nature as some of the fuzzy filter parameters (not all) are random variables.
- The inverse-variance is modeled as a Gamma distribution.
- The parameters of the stochastic model of heartbeat intervals must be inferred with the data of R-R interval series. That is, the following problems must be solved:
  - posterior probability inference of the inverse-variance of heartbeat interval distribution,
  - posterior probability inference of the random parameters of fuzzy filter,
  - estimation of the deterministic parameters of fuzzy filter.

  These problems can be solved under VB paradigm.
- The individual and physiological condition specific heartbeat interval model is inferred with the corresponding data of R-R interval series.

- Once the different heartbeat interval models (such that each model describe a particular physiological state of an individual) are available, an analysis of the given R-R interval series generated under an unknown physiological state is performed as follows:
  - The given R-R interval series is assumed to be generated by a stochastic mixture of a finite number of physiological state specific models such that each model tries to fit a part of the series.
  - The parameters of the mixture are inferred using VB algorithm. The estimated parameters may serve as the features of the series. Further, the posterior probabilities of a data point being generated by different physiological state specific models can be compared to predict the most probable physiological state.

The aforementioned approach to heartbeat intervals analysis is mathematically a special case of the problem of "fuzzy filters mixture inference". Therefore, we provide independently the details of the problem of VB inference of the fuzzy filters mixture. This problem is closely related to the work of [17] with the difference that an individual filter, instead of trying to model the total data set, tries to model a part of the data. To the best knowledge of the authors, the application of fuzzy filters mixture in the stochastic modeling of heartbeat intervals is the novelty of the work. It is demonstrated in a tilt-tale experiment on 40 healthy subjects that

1. the extracted features of heartbeat intervals are relevant to the physiological state of the individual,
2. the accuracy of the predicted physiological state, as a part of the feature extraction procedure, is good.

This text is organized into sections. Section 2 provides the essential background for the method. The VB inference problem of fuzzy filters mixture is discussed in section 3. Section 4 present a fuzzy filtering based stochastic modeling of heartbeat intervals followed by the details of our analysis method in section 5. The experimental studies are presented in section 6 and finally the concluding remarks in section 7.

2 Preliminary

2.1 Variational Bayesian Inference

Bayesian framework based on Bayes' theorem is a powerful technique for the statistical inference of model parameters. The VB framework has the advantage of being less computationally intensive than other Bayesian methods. The VB method approximates the posterior distributions over a model in an analytical manner [3]. The VB method minimizes the Kullback-Leibler (KL) divergence of the approximate posterior from the true posterior density [25].

The Bayesian inference problem determines the parameters w of a model $\mathfrak{m}$ using available data y based on Bayes's theorem:

$$p(w|y,\mathfrak{m}) = \frac{p(y|w,\mathfrak{m})p(w|\mathfrak{m})}{p(y|\mathfrak{m})}.$$

Bayes's theorem provides the *posterior* probability of the parameters given the data and the model. For a model, it may not be possible an analytical evaluation of posterior probability distribution. Thus, it is approximated by a variational distribution:

$$q(w) \approx p(w|y, \mathfrak{m})$$

where $q(w)$ is restricted to belong to a family of distributions of simpler form. This form is selected by minimizing the difference (in terms of Kullback-Leibler divergence) between q and true posterior. The Kullback-Leibler (KL) divergence of $p(w|y, \mathfrak{m})$ from $q(w)$ is defined as

$$KL(q||p) = \int q(w) \log \frac{q(w)}{p(w|y, \mathfrak{m})} dw.$$

The logarithmic *evidence* for the data is given as

$$\begin{aligned}
\log p(y|\mathfrak{m}) &= \log \int p(y, w|\mathfrak{m}) dw \\
&= \log \int q(w) \frac{p(y, w|\mathfrak{m})}{q(w)} dw \\
&\geq \int q(w) \log \frac{p(y, w|\mathfrak{m})}{q(w)} dw \\
&\equiv \mathcal{F}(q(w), \mathfrak{m})
\end{aligned}$$

where we have made use of the Jensen's inequality. Any probability distribution $q(w)$ gives rise to a lower bound $\mathcal{F}(q(w), \mathfrak{m})$ on the logarithmic evidence. The lower bound $\mathcal{F}(q(w), \mathfrak{m})$ is the negative of a quantity known as *free energy*. Since

$$\log p(y|\mathfrak{m}) = \mathcal{F}(q(w), \mathfrak{m}) + KL(q||p),$$

minimizing $KL(q||p)$ is equivalent to maximizing $\mathcal{F}(q(w), \mathfrak{m})$ over $q(w)$. Therefore, posterior distribution $p(w|y, \mathfrak{m})$ is inferred by estimating $q(w)$ correctly, i.e., by maximizing $\mathcal{F}(q(w), \mathfrak{m})$ over $q(w)$.

2.2 Variational Bayes for a Mixed Stochastic/Deterministic Fuzzy Filter [17]

A Takagi-Sugeno fuzzy filter, as explained in Appendix A.1, can be mathematically represented as

$$y_f = G^T(x, \theta)\alpha, \ c\theta \geq h. \tag{1}$$

The filter, as seen from (1), is characterized by two types of parameters: antecedents (θ) and consequents (α). Expression (1) shows that the output of the fuzzy filter is linear in consequents (i.e. in the elements of vector α) while nonlinear in antecedents (i.e. in the elements of vector θ). We study a type of filter with

- the nonlinear parameters θ being considered as deterministic, and
- the linear parameters α being considered as random variables.

We focus on a process with $n-$inputs (represented by the vector $x \in R^n$) and a single output (represented by the scalar y). It is assumed that inputs-output data pairs $\{x(j), y(j)\}$ are related via

$$y(j) = G^T(x(j), \theta)\alpha + n_j, \tag{2}$$

where n_j is the additive Gaussian uncertainty with mean 0 and a variance of $1/\phi$. Given N pairs of inputs-output data $\{x(j), y(j)\}_{j=1}^N$, the fuzzy filtering algorithms should seek to estimate the vector θ and evaluate the posterior probability distribution of α. Introduce the notations:

$$Y = [y(1) \cdots y(N)]^T \in R^N, \; B(\theta) = \begin{bmatrix} G^T(x(1), \theta) \\ \vdots \\ G^T(x(N), \theta) \end{bmatrix} \in R^{N \times K}, \; v = [n_1 \cdots n_N]^T \in R^N.$$

Now, we have

$$Y = B(\theta)\alpha + v, \tag{3}$$

where v is an additive Gaussian uncertainty with mean 0 and a variance of $1/\phi$:

$$p(v) \sim N(0, \phi^{-1} I).$$

The considered fuzzy filtering problem in the VB framework is stated in Problem 1.

Problem 1 *Given N pairs of inputs-output data $\{x(j), y(j)\}_{j=1}^N$ and a structure $\mathfrak{m}$ (i.e. membership type, number of membership functions and rules) of a Takagi-Sugeno filter of type (1) such that data satisfy (3), estimate θ and the variational distributions $(q(\alpha), q(\phi))$ by maximizing the lower bound on the quantity:* $\log p(Y|B(\theta), \mathfrak{m})$.

Problem 1 was solved in [17] with the following parameters priors:

$$p(\alpha|m_0, \Lambda_0) = N(\alpha|m_0, (\Lambda_0)^{-1})$$
$$p(\phi|a_0, b_0) = Ga(\phi|a_0, b_0)$$

where Gamma distribution is defined as follows

$$Ga(\phi|a_0, b_0) = \frac{1}{\Gamma(b_0)} \frac{\phi^{b_0 - 1}}{a_0^{b_0}} e^{-\frac{\phi}{a_0}}, \text{ for } \phi > 0 \text{ and } a_0, b_0 > 0.$$

The algorithm stated in [17] to estimate θ^* and to infer the variational distributions, i.e. $q^*(\alpha) = N(\alpha|m, (\Lambda)^{-1})$ and $q^*(\phi) = Ga(\phi|a, b)$, is repeated here in Appendix A.2 for the sake of completion.

3 A Mixture of Fuzzy Filters

3.1 Problem Formulation

Assume that the given data set $\{x(j), y(j)\}_{j=1}^N$ has been generated by a stochastic mixture of S different fuzzy filters (with their structures as $\{\mathfrak{m}^s\}_{s=1}^S$ and parameters as $\{\alpha^s, \theta^s\}_{s=1}^S$) such that an individual filter tries to model a part of the data. Let $\{s_i\}_{i=1}^N$ denote N different discrete random variables such that the value of s_i (where $s_i = 1, 2, \cdots, S$) represents the chosen fuzzy filter for modeling the $i-$th pair of data $(x(i), y(i))$. That is, $\forall i = 1, \cdots, N$,

$$\begin{aligned} &\text{If } s_i = 1, \text{ then } y(i) = G^T(x(i), \theta^1)\alpha^1 + n_i, \; p(n_i) \sim N(0, (\phi^1)^{-1}) \\ &\quad \vdots \\ &\text{If } s_i = S, \text{ then } y(i) = G^T(x(i), \theta^S)\alpha^S + n_i, \; p(n_i) \sim N(0, (\phi^S)^{-1}) \end{aligned} \tag{4}$$

Introduce the notations:

$$Y = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix} \in R^N, \; B(\theta^s) = \begin{bmatrix} G^T(x(1), \theta^s) \\ \vdots \\ G^T(x(N), \theta^s) \end{bmatrix} \in R^{N \times K^s}.$$

Let $\pi = [\pi_1 \cdots \pi_S]^T \in R^S$, with $0 \leq \pi_{s_i} \leq 1$ and $\sum_{s_i=1}^{S} \pi_{s_i} = 1$, be a vector of mixing proportions (i.e. the proportions by which individual fuzzy filters' outputs are mixed to match the observed output vector) such that the discrete distribution of s_i is given as

$$p(s_i = 1|\pi) = \pi_1, \; \cdots, \; p(s_i = S|\pi) = \pi_S.$$

The probability of an i.i.d. data set Y is given as

$$\begin{aligned} & p(Y|\pi, \{B(\theta^s)\}_{s=1}^S, \{\alpha^s\}_{s=1}^S, \{\phi^s\}_{s=1}^S, \{\mathfrak{m}^s\}_{s=1}^S) \\ & = \prod_{i=1}^{N} p(y(i)|\pi, \{G(x(i), \theta^s)\}_{s=1}^S, \{\alpha^s\}_{s=1}^S, \{\phi^s\}_{s=1}^S, \{\mathfrak{m}^s\}_{s=1}^S) \\ & = \prod_{i=1}^{N} \sum_{s=1}^{S} p(s_i = s|\pi) p(y(i)|s_i = s, G(x(i), \theta^s), \alpha^s, \phi^s, \mathfrak{m}^s) \end{aligned}$$

Following distributions are chosen for the parameters priors:

$$\begin{aligned} p(\alpha^s|m_0^s, \mathit{\Lambda}_0^s) &= N(\alpha^s|m_0^s, (\mathit{\Lambda}_0^s)^{-1}) \\ p(\phi^s|a_0^s, b_0^s) &= Ga(\phi^s|a_0^s, b_0^s) \\ p(\pi|c_0 d_0) &= Dir(\pi|c_0 d_0), \text{ such that } d_0 = [\tfrac{1}{S} \cdots \tfrac{1}{S}]^T \in R^S \end{aligned}$$

where Gamma and Dirichlet distributions are defined as follows

$$Ga(\phi^s|a_0^s, b_0^s) = \frac{1}{\mathit{\Gamma}(b_0^s)} \frac{(\phi^s)^{b_0^s - 1}}{(a_0^s)^{b_0^s}} e^{-\dfrac{\phi^s}{a_0^s}}, \text{ for } \phi^s > 0 \text{ and } a_0^s, b_0^s > 0.$$

$$Dir(\pi|c_0 d_0) = \frac{\mathit{\Gamma}(c_0)}{(\mathit{\Gamma}(\frac{c_0}{S}))^S} \pi_1^{\dfrac{c_0}{S} - 1} \cdots \pi_S^{\dfrac{c_0}{S} - 1}, \text{ where } \pi_1, \cdots, \pi_S \geq 0, \; \sum_{j=1}^{S} \pi_j = 1, \; c_0 > 0.$$

The logarithmic evidence for the data is given as

$$\begin{aligned} & \log p(Y|\{B(\theta^s)\}_{s=1}^S, \{\mathfrak{m}^s\}_{s=1}^S) = \log \int d\pi d\alpha^1 \cdots d\alpha^S d\phi^1 \cdots d\phi^S p(\pi|c_0 d_0) p(\alpha^1|m_0^1, \mathit{\Lambda}_0^1) \cdots \\ & \cdots p(\alpha^S|m_0^S, \mathit{\Lambda}_0^S) p(\phi^1|a_0^1, b_0^1) \cdots p(\phi^S|a_0^S, b_0^S) p(Y|\pi, \{B(\theta^s)\}_{s=1}^S, \{\alpha^s\}_{s=1}^S, \{\phi^s\}_{s=1}^S, \{\mathfrak{m}^s\}_{s=1}^S). \end{aligned}$$

For simplicity, define $\alpha = \{\alpha^1, \cdots, \alpha^S\}$, $m_0 = \{m_0^1, \cdots, m_0^S\}$, $\mathit{\Lambda}_0 = \{\mathit{\Lambda}_0^1, \cdots, \mathit{\Lambda}_0^S\}$, $\phi = \{\phi^1, \cdots, \phi^S\}$, $a_0 = \{a_0^1, \cdots, a_0^S\}$, $b_0 = \{b_0^1, \cdots, b_0^S\}$ with

$$p(\alpha|m_0, \mathit{\Lambda}_0) = \prod_{s=1}^{S} p(\alpha^s|m_0^s, \mathit{\Lambda}_0^s), \; p(\phi|a_0, b_0) = \prod_{s=1}^{S} p(\phi^s|a_0^s, b_0^s).$$

Rewriting the data evidence as

$$\log p(Y|\{B(\theta^s)\}_{s=1}^S, \{\mathfrak{m}^s\}_{s=1}^S) =$$
$$\log \int d\pi d\alpha d\phi p(\pi|c_0 d_0) p(\alpha|m_0, \Lambda_0) p(\phi|a_0, b_0) \underbrace{\prod_{i=1}^N \left\{ \sum_{s=1}^S p(s_i = s|\pi) p(y(i)|s_i = s, G(x(i), \theta^s), \alpha^s, \phi^s, \mathfrak{m}^s) \right\}}_{= p(Y|\pi, \{B(\theta^s)\}_{s=1}^S, \alpha, \phi, \{\mathfrak{m}^s\}_{s=1}^S)}.$$

Any arbitrary distribution $q(\pi, \alpha, \phi)$ gives rise to a lower bound on the data evidence:

$$\log p(Y|\{B(\theta^s)\}_{s=1}^S, \{\mathfrak{m}^s\}_{s=1}^S) \geq \int d\pi d\alpha d\phi q(\pi, \alpha, \phi)$$
$$\log \frac{p(\pi|c_0 d_0) p(\alpha|m_0, \Lambda_0) p(\phi|a_0, b_0) p(Y|\pi, \{B(\theta^s)\}_{s=1}^S, \alpha, \phi, \{\mathfrak{m}^s\}_{s=1}^S)}{q(\pi, \alpha, \phi)}.$$

Restricting the arbitrary distribution $q(\pi, \alpha, \phi)$ with the following approximation

$$q(\pi, \alpha, \phi) \approx q(\pi) q(\alpha) q(\phi).$$

This results in

$$\log p(Y|\{B(\theta^s)\}_{s=1}^S, \{\mathfrak{m}^s\}_{s=1}^S) \geq$$
$$\int d\pi q(\pi) \log \frac{p(\pi|c_0 d_0)}{q(\pi)} + \int d\alpha q(\alpha) \log \frac{p(\alpha|m_0, \Lambda_0)}{q(\alpha)} + \int d\phi q(\phi) \log \frac{p(\phi|a_0, b_0)}{q(\phi)}$$
$$+ \int d\pi d\alpha d\phi q(\pi) q(\alpha) q(\phi) \log(p(Y|\pi, \{B(\theta^s)\}_{s=1}^S, \alpha, \phi, \{\mathfrak{m}^s\}_{s=1}^S)).$$

In other words,

$$\log p(Y|\{B(\theta^{s_i})\}_{s_i=1}^S, \{\mathfrak{m}^{s_i}\}_{s_i=1}^S) \geq$$
$$\int d\pi q(\pi) \log \frac{p(\pi|c_0 d_0)}{q(\pi)} + \int d\alpha q(\alpha) \log \frac{p(\alpha|m_0, \Lambda_0)}{q(\alpha)} + \int d\phi q(\phi) \log \frac{p(\phi|a_0, b_0)}{q(\phi)}$$
$$+ \int d\pi d\alpha d\phi q(\pi) q(\alpha) q(\phi) \sum_{i=1}^N \log(\sum_{s=1}^S p(s_i = s|\pi) p(y(i)|s_i = s, G(x(i), \theta^s), \alpha^s, \phi^s, \mathfrak{m}^s)).$$

Again, an arbitrary discrete distribution $q(s_i)$ with $\sum_{s=1}^S q(s_i = s) = 1$, is introduced to further lower bound the data evidence:

$$\log p(Y|\{B(\theta^s)\}_{s=1}^S, \{\mathfrak{m}^s\}_{s=1}^S) \geq$$
$$\int d\pi q(\pi) \log \frac{p(\pi|c_0 d_0)}{q(\pi)} + \int d\alpha q(\alpha) \log \frac{p(\alpha|m_0, \Lambda_0)}{q(\alpha)} + \int d\phi q(\phi) \log \frac{p(\phi|a_0, b_0)}{q(\phi)}$$
$$+ \int d\pi d\alpha d\phi q(\pi) q(\alpha) q(\phi) \sum_{i=1}^N \sum_{s=1}^S q(s_i = s) \log \frac{p(s_i = s|\pi) p(y(i)|s_i = s, G(x(i), \theta^s), \alpha^s, \phi^s, \mathfrak{m}^s)}{q(s_i = s)}.$$

That is,

$$\log p(Y|\{B(\theta^s)\}_{s=1}^S, \{\mathfrak{m}^s\}_{s=1}^S) \geq$$
$$\int d\pi q(\pi) \log \frac{p(\pi|c_0 d_0)}{q(\pi)} + \int d\alpha q(\alpha) \log \frac{p(\alpha|m_0, \Lambda_0)}{q(\alpha)} + \int d\phi q(\phi) \log \frac{p(\phi|a_0, b_0)}{q(\phi)}$$

$$+\sum_{i=1}^{N}\left\{\sum_{s=1}^{S} q(s_i=s)\int d\pi q(\pi)\log\frac{p(s_i=s|\pi)}{q(s_i=s)}\right.$$
$$\left.+\sum_{s=1}^{S} q(s_i=s)\int d\alpha d\phi q(\alpha)q(\phi)\log p(y(i)|s_i=s,G(x(i),\theta^s),\alpha^s,\phi^s,\mathfrak{m}^s)\right\}.$$

The lower bound is defined as a functional of the variational posterior distributions as follows:

$$\mathcal{F}(q(\pi),q(\alpha),q(\phi),\{q(s_i)\}_{i=1}^{N},\{B(\theta^s)\}_{s=1}^{S},c_0,m_0,\Lambda_0,a_0,b_0,\{\mathfrak{m}^s\}_{s=1}^{S})=$$
$$\int d\pi q(\pi)\log\frac{p(\pi|c_0d_0)}{q(\pi)}+\int d\alpha q(\alpha)\log\frac{p(\alpha|m_0,\Lambda_0)}{q(\alpha)}+\int d\phi q(\phi)\log\frac{p(\phi|a_0,b_0)}{q(\phi)}$$
$$+\sum_{i=1}^{N}\left\{\sum_{s=1}^{S} q(s_i=s)\int d\pi q(\pi)\log\frac{p(s_i=s|\pi)}{q(s_i=s)}\right.$$
$$\left.+\sum_{s=1}^{S} q(s_i=s)\int d\alpha d\phi q(\alpha)q(\phi)\log p(y(i)|s_i=s,G(x(i),\theta^s),\alpha^s,\phi^s,\mathfrak{m}^s)\right\}.$$

Since

$$\log p(y(i)|s_i=s,G(x(i),\theta^s),\alpha^s,\phi^s,\mathfrak{m}^s)=-\frac{1}{2}\log(2\text{pi})+\frac{1}{2}\log(\phi^s)-\frac{\phi^s}{2}|y(i)-G^T(x(i),\theta^s)\alpha^s|^2,$$

therefore

$$\mathcal{F}(q(\pi),q(\alpha),q(\phi),\{q(s_i)\}_{i=1}^{N},\{B(\theta^s)\}_{s=1}^{S},c_0,m_0,\Lambda_0,a_0,b_0,\{\mathfrak{m}^s\}_{s=1}^{S})=$$
$$\int d\pi q(\pi)\log\frac{p(\pi|c_0d_0)}{q(\pi)}+\int d\alpha q(\alpha)\log\frac{p(\alpha|m_0,\Lambda_0)}{q(\alpha)}+\int d\phi q(\phi)\log\frac{p(\phi|a_0,b_0)}{q(\phi)}-\frac{N}{2}\log(2\text{pi})$$
$$+\sum_{i=1}^{N}\left\{\sum_{s=1}^{S} q(s_i=s)\int d\pi q(\pi)\log\frac{p(s_i=s|\pi)}{q(s_i=s)}+\frac{1}{2}\sum_{s=1}^{S} q(s_i=s)\int d\phi^s q(\phi^s)\log(\phi^s)\right.$$
$$\left.-\frac{1}{2}\sum_{s=1}^{S} q(s_i=s)\int d\alpha^s d\phi^s q(\alpha^s)q(\phi^s)\phi^s|y(i)-G^T(x(i),\theta^s)\alpha^s|^2\right\}.$$

Problem 2 *Given N pairs of inputs-output data* $\{x(j),y(j)\}_{j=1}^{N}$ *and S different structures* $\{\mathfrak{m}^s\}_{s=1}^{S}$ *of the Takagi-Sugeno filters of type (1) such that data satisfy (4), estimate* $\{\theta^s\}_{s=1}^{S}$ *and the variational distributions* $\left(\{q(\alpha^s)\}_{s=1}^{S},\{q(s_i)\}_{i=1}^{N},q(\pi),\{q(\phi^s)\}_{s=1}^{S}\right)$ *by maximizing the lower bound on the quantity:* $\log p(Y|\{B(\theta^s)\}_{s=1}^{S},\{\mathfrak{m}^s\}_{s=1}^{S})$.

Remark 1 Please note the difference between Problem 2 and the main problem of [17]. Here, an individual filter tries to model a part of the data unlike the problem of [17] where an individual filter tries to model the total data set.

3.2 Optimization of the Lower Bound $\mathcal{F}$

3.2.1 Optimization w.r.t. $q(\pi)$

$\mathcal{F}$ will be stationary w.r.t. distribution $q(\pi)$, if

$$\log(p(\pi|c_0d_0))-\log(q(\pi))+\sum_{i=1}^{N}\sum_{s=1}^{S} q(s_i=s)\log(p(s_i=s|\pi))+\text{cons}\{q(\pi)\}=0,\ \text{i.e.,}$$

$$\log(\prod_{s=1}^{S} \pi_s^{\frac{c_0}{S}-1}) - \log(q(\pi)) + \sum_{i=1}^{N}\sum_{s=1}^{S} \log(\pi_s^{q(s_i=s)}) + \text{cons}\{q(\pi)\} = 0, \text{ i.e.,}$$

$$\log(\prod_{s=1}^{S} \pi_s^{\frac{c_0}{S}-1}) - \log(q(\pi)) + \log(\prod_{s=1}^{S} \pi_s^{\sum_{i=1}^{N} q(s_i=s)}) + \text{cons}\{q(\pi)\} = 0, \text{ i.e.,}$$

$$\log(\prod_{s=1}^{S} \pi_s^{\frac{c_0}{S}+\sum_{i=1}^{N} q(s_i=s)-1}) - \log(q(\pi)) + \text{cons}\{q(\pi)\} = 0.$$

This implies that

$$q^*(\pi) = Dir(\pi|cd), \ d = [d_1 \cdots d_S]^T \in R^S \text{ with } \textstyle\sum_{s=1}^{S} d_s = 1,$$

such that

$$cd_s = \frac{c_0}{S} + \sum_{i=1}^{N} q(s_i = s).$$

Using $\sum_{s=1}^{S} d_s = 1$ and $\sum_{s=1}^{S} q(s_i = s) = 1$, we have

$$c = c_0 + N.$$

Thus,

$$d_s = \frac{1}{c_0 + N}\left(\frac{c_0}{S} + \sum_{i=1}^{N} q(s_i = s)\right), \ s = 1, \cdots, S.$$

3.2.2 Optimization w.r.t. $q(\alpha^1), \cdots, q(\alpha^S)$

$\mathcal{F}$ will be stationary w.r.t. distribution $q(\alpha^s)$ where $s = 1, \cdots, S$, if

$$\begin{aligned}&\log(p(\alpha^s|m_0^s, \Lambda_0^s)) - \log(q(\alpha^s))\\ &-\frac{\left(\int d\phi^s q(\phi^s)\phi^s\right)}{2}\sum_{i=1}^{N} q(s_i = s)|y(i) - G^T(x(i), \theta^s)\alpha^s|^2 + cons\{q(\alpha^s)\} = 0.\end{aligned}$$

The substitution

$$p(\alpha^s|m_0^s, \Lambda_0^s) \propto e^{-\frac{1}{2}(\alpha^s - m_0^s)^T \Lambda_0^s(\alpha^s - m_0^s)},$$

results in

$$\begin{aligned}&-\frac{1}{2}(\alpha^s - m_0^s)^T \Lambda_0^s(\alpha^s - m_0^s) - \log(q(\alpha^s))\\ &-\frac{\left(\int d\phi^s q(\phi^s)\phi^s\right)}{2}\sum_{i=1}^{N} q(s_i = s)|y(i) - G^T(x(i), \theta^s)\alpha^s|^2 + cons\{q(\alpha^s)\} = 0.\end{aligned}$$

This implies that

$$q^*(\alpha^s) = N(\alpha^s|m^s, (\Lambda^s)^{-1}), \text{ such that}$$

$$\Lambda^s = \Lambda_0^s + \left(\int d\phi^s q(\phi^s)\phi^s \right) (B_q(\theta^s))^T B_q(\theta^s),$$

$$\Lambda^s m^s = \Lambda_0^s m_0^s + \left(\int d\phi^s q(\phi^s)\phi^s \right) (B_q(\theta^s))^T Y_q^s,$$

$$B_q(\theta^s) = \begin{bmatrix} \sqrt{q(s_1 = s)} G^T(x(1), \theta^s) \\ \vdots \\ \sqrt{q(s_N = s)} G^T(x(N), \theta^s) \end{bmatrix} \in R^{N \times K^s}, \; Y_q^s = \begin{bmatrix} \sqrt{q(s_1 = s)} y(1) \\ \vdots \\ \sqrt{q(s_N = s)} y(N) \end{bmatrix} \in R^N.$$

The term $\int d\phi^s q(\phi^s)\phi^s$, appearing in the expressions for Λ^s and m^s, will be evaluated after obtaining the correct expression for $q(\phi^s)$.

3.2.3 Optimization w.r.t. $q(\phi^1), \cdots, q(\phi^S)$

$\mathcal{F}$ will be stationary w.r.t. distribution $q(\phi^s)$, if

$$\log(p(\phi^s|a_0^s, b_0^s)) - \log(q(\phi^s)) + cons\{\phi^s\} + \sum_{i=1}^{N} \left\{ \frac{q(s_i = s)}{2} \log(\phi^s) - \frac{q(s_i = s)}{2} \phi^s \int d\alpha^s q(\alpha^s) |y(i) - G^T(x(i), \theta^s)\alpha^s|^2 \right\} = 0, \text{ i.e.,}$$

$$\log(p(\phi^s|a_0^s, b_0^s)) - \log(q(\phi^s)) + cons\{\phi^s\} + \frac{\sum_{i=1}^N q(s_i = s)}{2} \log(\phi^s) - \frac{\phi^s}{2} \sum_{i=1}^{N} q(s_i = s) \int d\alpha^s q(\alpha^s) |y(i) - G^T(x(i), \theta^s)\alpha^s|^2 = 0, \text{ i.e.,}$$

$$\log(p(\phi^s|a_0^s, b_0^s)) - \log(q(\phi^s)) + cons\{\phi^s\} + \frac{\sum_{i=1}^N q(s_i = s)}{2} \log(\phi^s) - \frac{\phi^s}{2} \int d\alpha^s q(\alpha^s) (Y_q^s - B_q(\theta^s)\alpha^s)^T (Y_q^s - B_q(\theta^s)\alpha^s) = 0.$$

Using the facts

$$\log(p(\phi^s|a_0^s, b_0^s)) = (b_0^s - 1)\log(\phi^s) - \frac{\phi^s}{a_0^s} + cons\{\phi^s\},$$

$$\int d\alpha^s q(\alpha^s) (Y_q^s - B_q(\theta^s)\alpha^s)^T (Y_q^s - B_q(\theta^s)\alpha^s) = (Y_q^s - B_q(\theta^s)m^s)^T (Y_q^s - B_q(\theta^s)m^s) + Tr\left((\Lambda^s)^{-1} (B_q(\theta^s))^T B_q(\theta^s) \right),$$

we have

$$(b_0^s - 1)\log(\phi^s) - \frac{\phi^s}{a_0^s} - \log(q(\phi^s)) + cons\{\phi^s\} + \frac{\sum_{i=1}^N q(s_i = s)}{2} \log(\phi^s) - \frac{\phi^s}{2} (Y_q^s - B_q(\theta^s)m^s)^T (Y_q^s - B_q(\theta^s)m^s) + Tr\left((\Lambda^s)^{-1} (B_q(\theta^s))^T B_q(\theta^s) \right) = 0.$$

This implies that

$$q^*(\phi^s) = Ga(\phi^s|a^s, b^s), \text{ such that}$$

$$b^s = b_0^s + \frac{\sum_{i=1}^N q(s_i = s)}{2},$$

$$\frac{1}{a^s} = \frac{1}{a_0^s} + \frac{1}{2} \left[(Y_q^s - B_q(\theta^s)m^s)^T (Y_q^s - B_q(\theta^s)m^s) + Tr\left((\Lambda^s)^{-1} (B_q(\theta^s))^T B_q(\theta^s) \right) \right].$$

3.2.4 Optimization w.r.t. $q(s_i = s)$

$\mathcal{F}$ will be stationary w.r.t. distribution $q(s_i = s)$, if

$$\int d\pi q(\pi)\log(p(s_i = s|\pi)) - \log(q(s_i = s)) \\ +\frac{1}{2}\int d\phi^s q(\phi^s)\log(\phi^s) - \frac{1}{2}\int d\alpha^s d\phi^s q(\alpha^s)q(\phi^s)\phi^s|y(i) - G^T(x(i),\theta^s)\alpha^s|^2 = 0$$

As per a result of Dirichlet distributions,

$$\int d\pi q(\pi)\log(p(s_i = s|\pi)) = \Psi(cd_s) - \Psi(c),$$

where $\Psi(\cdot)$ is the digamma function. Now, consider the term

$$\int d\alpha^s d\phi^s q(\alpha^s)q(\phi^s)\phi^s|y(i) - G^T(x(i),\theta^s)\alpha^s|^2 = \\ \left[|y(i) - G^T(x(i),\theta^s)m^s|^2 + Tr\left((\Lambda^s)^{-1}G(x(i),\theta^s)G^T(x(i),\theta^s)\right)\right]\int d\phi^s q(\phi^s)\phi^s.$$

Using

$$\int d\phi^s q(\phi^s)\phi^s = a^s b^s, \text{ and } \int d\phi^s q(\phi^s)\log(\phi^s) = \Psi(b^s) + \log(a^s),$$

the equilibrium equation becomes

$$\Psi(cd_s) - \Psi(c) - \log(q(s_i = s)) + \frac{1}{2}(\Psi(b^s) + \log(a^s)) \\ -\frac{a^s b^s}{2}\left[|y(i) - G^T(x(i),\theta^s)m^s|^2 + Tr\left((\Lambda^s)^{-1}G(x(i),\theta^s)G^T(x(i),\theta^s)\right)\right] = 0.$$

This implies that

$$q^*(s_i = s) = \frac{1}{\mathcal{Z}}e^{\Psi(cd_s) - \Psi(c) + \frac{1}{2}(\Psi(b^s) + \log(a^s)) - \frac{a^s b^s}{2}r_i(m^s,\Lambda^s,\theta^s)},$$

where $\mathcal{Z}$ is the normalization constant such that $\sum_{s=1}^{S} q^*(s_i = s) = 1$ and

$$r_i(m^s,\Lambda^s,\theta^s) = |y(i) - G^T(x(i),\theta^s)m^s|^2 + Tr\left((\Lambda^s)^{-1}G(x(i),\theta^s)G^T(x(i),\theta^s)\right).$$

3.2.5 Optimization w.r.t. $(\theta^1,\cdots,\theta^S)$

The optimal values of antecedents of fuzzy filters are obtained via maximizing

$$\mathcal{F}(q^*(\pi),q^*(\alpha),q^*(\phi),\{q^*(s_i)\}_{i=1}^{N},\{B(\theta^s)\}_{s=1}^{S},c_0,m_0,\Lambda_0,a_0,b_0,\{\mathfrak{m}^s\}_{s=1}^{S})$$

over $(\theta^1, \cdots, \theta^S)$. The lower bound on the logarithmic evidence for the data can be expressed as

$$\mathcal{F}(q(\pi), q(\alpha), q(\phi), \{q(s_i)\}_{i=1}^N, \{B(\theta^s)\}_{s=1}^S, c_0, m_0, \Lambda_0, a_0, b_0, \{\mathfrak{m}^s\}_{s=1}^S) =$$
$$\sum_{i=1}^N \left\{ \sum_{s=1}^S q(s_i = s) \int d\pi q(\pi) \log \frac{p(s_i = s|\pi)}{q(s_i = s)} + \frac{1}{2} \sum_{s=1}^S q(s_i = s) \int d\phi^s q(\phi^s) \log(\phi^s) \right.$$
$$\left. -\frac{1}{2} \sum_{s=1}^S q(s_i = s) \int d\alpha^s d\phi^s q(\alpha^s) q(\phi^s) \phi^s |y(i) - G^T(x(i), \theta^s)\alpha^s|^2 \right\} - \frac{N}{2} \log(2\text{pi})$$
$$- \int d\pi q(\pi) \log \frac{q(\pi)}{p(\pi|c_0 d_0)} - \int d\alpha q(\alpha) \log \frac{q(\alpha)}{p(\alpha|m_0, \Lambda_0)} - \int d\phi q(\phi) \log \frac{q(\phi)}{p(\phi|a_0, b_0)}.$$

The value of $\mathcal{F}$ at obtained optimal distributions (i.e. at $q(\pi) = q^*(\pi), q(\alpha) = q^*(\alpha), q(\phi) = q^*(\phi), q(s_i) = q^*(s_i)$) is given as

$$\mathcal{F}(q^*(\pi), q^*(\alpha), q^*(\phi), \{q^*(s_i)\}_{i=1}^N, \{B(\theta^s)\}_{s=1}^S, c_0, m_0, \Lambda_0, a_0, b_0, \{\mathfrak{m}^s\}_{s=1}^S) = -\frac{N}{2} \log(2\text{pi})$$
$$+ \sum_{i=1}^N \left\{ \sum_{s=1}^S q^*(s_i = s) \left[\Psi(cd_s) - \Psi(c) - \log(q(s_i = s))\right] + \frac{1}{2} \sum_{s=1}^S q^*(s_i = s) \left[\Psi(b^s) + \log(a^s)\right] \right.$$
$$\left. -\frac{1}{2} \sum_{s=1}^S q^*(s_i = s) a^s b^s \left[|y(i) - G^T(x(i), \theta^s) m^s|^2 + Tr\left((\Lambda^s)^{-1} G(x(i), \theta^s) G^T(x(i), \theta^s) \right) \right] \right\}$$
$$- \left[\log\left(\frac{\Gamma(c)}{\Gamma(c_0)} \right) + \sum_{s=1}^S \log\left(\frac{\Gamma(c_0/S)}{\Gamma(cd_s)} \right) + \sum_{s=1}^S \left[cd_s - \frac{c_0}{S} \right] \left[\Psi(cd_s) - \Psi(c)\right] \right]$$
$$- \sum_{s=1}^S \left[\frac{1}{2} \log\left(\frac{|(\Lambda_0^s)^{-1}|}{|(\Lambda^s)^{-1}|} \right) + \frac{1}{2} Tr\left(\Lambda_0^s (\Lambda^s)^{-1} \right) + \frac{1}{2} (m^s - m_0^s)^T \Lambda_0^s (m^s - m_0^s) - \frac{K^s}{2} \right]$$
$$- \sum_{s=1}^S \left[\log(\Gamma(b_0^s)) + b_0^s \log(a_0^s) - \log(\Gamma(b^s)) - b_0^s \log(a^s) + b^s \Psi(b^s) - b_0^s \Psi(b^s) + b^s \frac{a^s}{a_0^s} - b^s \right].$$

At this point, we observe that for the given values of probability mass functions $(\{q^*(s_i)\}_{i=1}^N)$ and parameters $(c, \{d_s\}_{s=1}^S, \{\Lambda^s\}_{s=1}^S, \{m^s\}_{s=1}^S, \{a^s\}_{s=1}^S, \{b^s\}_{s=1}^S)$, an increase in the value of $\mathcal{F}$ will occur if parameters $\{\theta^s\}_{s=1}^S$ are optimized based on the following optimization problem:

$$\theta^{s,*} = \arg \min_{\theta^s} \left[\sum_{i=1}^N q^*(s_i = s) r_i(m^s, \Lambda^s, \theta^s);\ c^s \theta^s \geq h^s \right],\ \forall s = 1, \cdots, S.$$

3.3 An Algorithm for a Finite Mixture of Fuzzy Filters

The rules for updating the parameters of the distributions and antecedents of fuzzy filters are summarized in the followings.

$$\theta^{s,*} = \arg \min_{\theta^s} \left[\sum_{i=1}^N q^*(s_i = s) r_i(m^s, \Lambda^s, \theta^s);\ c^s \theta^s \geq h^s \right]$$
$$\Lambda^s = \Lambda_0^s + a^s b^s (B_q(\theta^{s,*}))^T B_q(\theta^{s,*})$$

$$m^s = \left[\Lambda^s\right]^{-1}\left[\Lambda_0^s m_0^s + a^s b^s (B_q(\theta^{s,*}))^T Y_q^s\right]$$

$$q^*(s_i = s) = \frac{1}{\mathcal{Z}} e^{\Psi(cd_s) - \Psi(c) + \frac{1}{2}\left[\Psi(b^s) + \log(a^s)\right] - \frac{a^s b^s}{2} r_i(m^s, \Lambda^s, \theta^{s,*})}$$

$$b^s = b_0^s + \frac{\sum_{i=1}^N q^*(s_i = s)}{2}$$

$$\frac{1}{a^s} = \frac{1}{a_0^s} + \frac{1}{2}\left[(Y_q^s - B_q(\theta^{s,*})m^s)^T (Y_q^s - B_q(\theta^{s,*})m^s) + Tr\left((\Lambda^s)^{-1}(B_q(\theta^{s,*}))^T B_q(\theta^{s,*})\right)\right]$$

$$c = c_0 + N$$

$$d_s = \frac{1}{c_0 + N}\left(\frac{c_0}{S} + \sum_{i=1}^N q^*(s_i = s)\right)$$

Here, in the expressions for Λ^s and m^s, the term $\int d\phi^s q(\phi^s)\phi^s$ has been substituted as $a^s b^s$.

Each of these update rules is guaranteed to monotonically increase the objective function $\mathcal{F}$. Therefore, several iterations of update rules can be performed to increase $\mathcal{F}$ until a consistent solution is reached. The iterative optimization process can be terminated if the maximum (over all possible i and s) change in $q^*(s_i = s)$ from an iteration to the next is less than a tolerance limit (say 0.0001). Algorithm 1 summarizes our method for inferring the parameters of the fuzzy filters mixture via maximizing $\mathcal{F}$.

4 Fuzzy Filtering based Heartbeat Interval Model

Given a short time recording of R-R intervals $\{RR_j\}_{j=1,2,\cdots}$ (where j is the time-index), consider a structure $\mathfrak{m}$ (i.e. membership type, number of membership functions and rules) of a Takagi-Sugeno filter of type (1) such that

$$RR_j = G^T([RR_{j-1}\ RR_{j-2} \cdots RR_{j-n}]^T, \theta)\alpha + n_j,$$

where n_j is the additive Gaussian uncertainty with mean 0 and a variance of $1/\phi$. Here, θ is deterministic while α and ϕ are random with the following priors:

$$p(\alpha|m_0, \Lambda_0) = N(\alpha|m_0, (\Lambda_0)^{-1})$$
$$p(\phi|a_0, b_0) = Ga(\phi|a_0, b_0)$$

Thus, the probability density of heartbeat intervals is given as

$$p(RR_j|RR_{j-1}, RR_{j-2}, \cdots, RR_{j-n}, \theta, \alpha, \phi, \mathfrak{m}) = \frac{1}{\sqrt{2\text{pi}(\phi)^{-1}}}\, e^{-\frac{\phi}{2}|RR_j - G^T([RR_{j-1}\ RR_{j-2} \cdots RR_{j-n}]^T, \theta)\alpha|^2}.$$

The estimation of θ and inference of approximate posterior distributions ($q(\alpha)$ and $q(\phi)$), under VB framework, is the problem formally stated in Problem 1. Thus, the algorithm 2 of Appendix A.2 can be used for inferring the parameters of heartbeat interval model. For an individual at a physiological state s, let $(\theta^s, \Lambda^s, m^s, a^s, b^s)$ denote the parameters returned by the algorithm 2. If there are S number of physiological states, then S different heartbeat interval models $\{(\theta^s, \Lambda^s, m^s, a^s, b^s)\}_{s=1}^S$ can be inferred.

Algorithm 1 An algorithm for VB inference of the fuzzy filters mixture

Require: Data pairs $\{x(j), y(j)\}_{j=1,\cdots,N}$.

1: Choose a total of S fuzzy filters' structures $\{\mathfrak{m}^s\}_{s=1}^S$; hyper-parameters $c_0, m_0, \Lambda_0, a_0, b_0$ which define the regularizing priors; parameters $\{c^s, h^s\}_{s=1}^S$ such that the interpretability constraints on the membership functions of the s-th filter can be formulated as $c^s\theta^s \geq h^s$.

2: Set iteration count $t = 0$ and choose a tolerance limit (say, equal to 0.01%).

3: **if** $\max_{(i,s)}\ abs\,(q^*(s_i = s)|_{t+1} - q^*(s_i = s)|_t) < 0.0001$ **then**

4: **return** $\{\theta^{s,*}|_{t+1}\}_{s=1}^S, c|_{t+1}, \{d_s|_{t+1}\}_{s=1}^S, \{\Lambda^s|_{t+1}\}_{s=1}^S, \{m^s|_{t+1}\}_{s=1}^S, \{q^*(s_i)|_{t+1}\}_{i=1}^N, \{a^s|_{t+1}\}_{s=1}^S, \{b^s|_{t+1}\}_{s=1}^S$.

5: **else**

6: Update, for $s = 1$ to S, the parameters as follows

$$\theta^{s,*}|_{t+1} = \arg\min_{\theta^s} \left[\sum_{i=1}^N q^*(s_i = s)|_t r_i(m^s|_t, \Lambda^s|_t, \theta^s);\ c^s\theta^s \geq h^s\right]$$

$$B_{q|_t}(\theta^{s,*}|_{t+1}) = \begin{bmatrix} \sqrt{q^*(s_1 = s)|_t}\,G^T(x(1), \theta^{s,*}|_{t+1}) \\ \vdots \\ \sqrt{q^*(s_N = s)|_t}\,G^T(x(N), \theta^{s,*}|_{t+1}) \end{bmatrix},\ Y^s_{q|_t} = \begin{bmatrix} \sqrt{q^*(s_1 = s)|_t}\,y(1) \\ \vdots \\ \sqrt{q^*(s_N = s)|_t}\,y(N) \end{bmatrix}$$

$$\Lambda^s|_{t+1} = \Lambda_0^s + a^s|_t b^s|_t (B_{q|_t}(\theta^{s,*}|_{t+1}))^T B_{q|_t}(\theta^{s,*}|_{t+1})$$

$$m^s|_{t+1} = [\Lambda^s|_{t+1}]^{-1}\left[\Lambda_0^s m_0^s + a^s|_t b^s|_t (B_{q|_t}(\theta^{s,*}|_{t+1}))^T Y^s_{q|_t}\right]$$

$$q^*(s_i = s)|_{t+1} = \frac{1}{\mathcal{Z}} e^{\Psi(c|_t d_s|_t) - \Psi(c|_t) + \frac{1}{2}[\Psi(b^s|_t) + \log(a^s|_t)] - \frac{a^s|_t b^s|_t}{2} r_i(m^s|_{t+1}, \Lambda^s|_{t+1}, \theta^{s,*}|_{t+1})}$$

$$b^s|_{t+1} = b_0^s + \frac{\sum_{i=1}^N q^*(s_i = s)|_{t+1}}{2}$$

$$\begin{aligned}\frac{1}{a^s|_{t+1}} = &\frac{1}{a_0^s} + \frac{1}{2}\Big[(Y^s_{q|_{t+1}} - B_{q|_{t+1}}(\theta^{s,*}|_{t+1})m^s|_{t+1})^T (Y^s_{q|_{t+1}} - B_{q|_{t+1}}(\theta^{s,*}|_{t+1})m^s|_{t+1}) \\ &+ Tr\left((\Lambda^s|_{t+1})^{-1}(B_{q|_{t+1}}(\theta^{s,*}|_{t+1}))^T B_{q|_{t+1}}(\theta^{s,*}|_{t+1})\right)\Big]\end{aligned}$$

$$c|_{t+1} = c_0 + N$$

$$d_s|_{t+1} = \frac{1}{c_0 + N}\left(\frac{c_0}{S} + \sum_{i=1}^N q^*(s_i = s)|_{t+1}\right)$$

Here, $(a|_0, b|_0, \{\theta^{s_i,*}|_0\}_{s_i=1}^S, \{q^*(s_i)|_0\}_{s_i=1}^S, c|_0, \{d_{s_i}|_0\}_{s_i=1}^S)$ denote an initial guess and

$$r_i(m^s, \Lambda^s, \theta^s) = |y(i) - G^T(x(i), \theta^s)m^s|^2 + Tr\left((\Lambda^s)^{-1} G(x(i), \theta^s) G^T(x(i), \theta^s)\right).$$

7: **end if**

5 Analysis of Heartbeat Intervals

Assume that the S number of heartbeat interval models for S different physiological states of an individual (i.e. $\{(\theta^s, \Lambda^s, m^s, a^s, b^s)\}_{s=1}^S$) have been inferred as explained in section 4. This section outlines the analysis of any R-R interval series generated under an unknown physiological state. Now, the given R-R interval series $\{RR_j\}_{j=1}^{N+n}$ is assumed to be generated by a stochastic mixture of S fuzzy filters (with the structures as $\{\mathfrak{m}^s\}_{s=1}^S$ and parameters as $\{\alpha^s, \theta^s\}_{s=1}^S$) such that an individual fuzzy filter tries to model a part of the series. Let $\{s_i\}_{i=1}^N$ denote N different discrete random variables such that the value of s_i (where $s_i = 1, 2, \cdots, S$) represents the chosen fuzzy filter for modeling the $i-$th pair of data $\left([RR_{i-1}\ RR_{i-2} \cdots RR_{i-n}]^T, RR_i\right)$. That is, $\forall i = 1, \cdots, N$,

$$\text{If } s_i = 1, \text{ then } RR_i = G^T([RR_{i-1}\ RR_{i-2} \cdots RR_{i-n}]^T, \theta^1)\alpha^1 + n_i,\ p(n_i) \sim N(0, (\phi^1)^{-1})$$
$$\vdots$$
$$\text{If } s_i = S, \text{ then } RR_i = G^T([RR_{i-1}\ RR_{i-2} \cdots RR_{i-n}]^T, \theta^S)\alpha^S + n_i,\ p(n_i) \sim N(0, (\phi^S)^{-1})$$

Introduce the notations:

$$Y = \begin{bmatrix} RR_{n+1} \\ \vdots \\ RR_{n+N} \end{bmatrix} \in R^N,\ B(\theta^s) = \begin{bmatrix} G^T([RR_n\ RR_{n-1} \cdots RR_1]^T, \theta^s) \\ \vdots \\ G^T([RR_{n+N-1}\ RR_{i-2} \cdots RR_N]^T, \theta^s) \end{bmatrix} \in R^{N \times K^s}.$$

Let $\pi = [\pi_1 \cdots \pi_S]^T \in R^S$, with $0 \leq \pi_{s_i} \leq 1$ and $\sum_{s_i=1}^{S} \pi_{s_i} = 1$, be a vector of mixing proportions (i.e. the proportions by which individual fuzzy filters' outputs are mixed to match the observed output vector) such that the discrete distribution of s_i is given as

$$p(s_i = 1|\pi) = \pi_1,\ \cdots,\ p(s_i = S|\pi) = \pi_S.$$

The probability of an i.i.d. data set Y is given as

$$\begin{aligned}
&p(Y|\pi, \{B(\theta^s)\}_{s=1}^S, \{\alpha^s\}_{s=1}^S, \{\phi^s\}_{s=1}^S, \{\mathfrak{m}^s\}_{s=1}^S) \\
&= \prod_{i=1}^{N} p(y(i)|\pi, \{B(\theta^s)\}_{s=1}^S, \{\alpha^s\}_{s=1}^S, \{\phi^s\}_{s=1}^S, \{\mathfrak{m}^s\}_{s=1}^S) \\
&= \prod_{i=1}^{N} \sum_{s=1}^{S} p(s_i = s|\pi) p(y(i)|s_i = s, \{B(\theta^s)\}_{s=1}^S, \{\alpha^s\}_{s=1}^S, \{\phi^s\}_{s=1}^S, \{\mathfrak{m}^s\}_{s=1}^S)
\end{aligned}$$

Following distributions are chosen for the parameters priors:

$$\begin{aligned}
p(\alpha^s|m_0^s, \Lambda_0^s) &= N(\alpha^s|m_0^s, (\Lambda_0^s)^{-1}) \\
p(\phi^s|a_0^s, b_0^s) &= Ga(\phi^s|a_0^s, b_0^s) \\
p(\pi|c_0 d_0) &= Dir(\pi|c_0 d_0),\ \text{such that } d_0 = [\tfrac{1}{S} \cdots \tfrac{1}{S}]^T \in R^S
\end{aligned}$$

where Gamma and Dirichlet distributions are defined as follows

$$Ga(\phi^s|a_0^s, b_0^s) = \frac{1}{\Gamma(b_0^s)} \frac{(\phi^s)^{b_0^s - 1}}{(a_0^s)^{b_0^s}} e^{-\dfrac{\phi^s}{a_0^s}},\ \text{for } \phi^s > 0 \text{ and } a_0^s, b_0^s > 0.$$

$$Dir(\pi|c_0 d_0) = \frac{\Gamma(c_0)}{(\Gamma(\frac{c_0}{S}))^S} \pi_1^{\dfrac{c_0}{S} - 1} \cdots \pi_S^{\dfrac{c_0}{S} - 1},\ \text{where } \pi_1, \cdots, \pi_S \geq 0,\ \sum_{j=1}^{S} \pi_j = 1,\ c_0 > 0.$$

Now, the estimation problem of $\{\theta^s\}_{s=1}^S, c, \{d_s\}_{s=1}^S, \{\Lambda^s\}_{s=1}^S, \{m^s\}_{s=1}^S, \{q(s_i)\}_{i=1}^N, \{a^s\}_{s=1}^S, \{b^s\}_{s=1}^S$ (stated formally in Problem 2) can be solved using algorithm 1.

Remark 2 The fuzzy filters mixture parameters returned by algorithm 1 are the features of heartbeat interval series. Further, the posterior distribution $q(s_i = s)$ provides the probability that the $i-$th data pair is generated under the physiological state s. This is how the physiological state is predicted as a part of the analysis method.

6 Experiments

We studied in total 40 subjects (21 men, 19 women, aged 22-45 years) during supine rest and 70° head-up tilt in the laboratory of Institute of Preventive Medicine, University of Rostock. All subjects were healthy without suffering from any heart related disease. The participants lay supine on the tilting table for 10 minutes followed by a 10 minutes head-up tilt at 70°. The physiological response of the subjects was assessed via measuring their heartbeat intervals (i.e. the time in milliseconds between consecutive R waves of an electrocardiogram). The R-R intervals were recorded using a Polar S-810i heart rate monitor (Polar Electro Oy, Finland).

Our first goal is to develop the heartbeat interval models for each of the two states where $s = 1$ corresponds to the physiological state of supine rest and $s = 2$ corresponds to the state of head-up tilt. For each of the two states, the initial and final half minute long data of R-R intervals were ignored and rest 9 minutes long data, divided into 3 segments with each segment 3 minutes long, were considered. The first 3 minutes long data segment was used to infer the fuzzy filtering based model of the physiological state using algorithm 2 and rest two segments were considered for an analysis using algorithm 1. However, for 6 out of 40 subjects, the 3 minutes long data segment was not sufficient to infer the reliable model of the physiological state. For these 6 subjects, for each state, the first 6 minutes long data were used for model inference and the rest 3 minutes long data for analysis.

Remark 3 The algorithm 1 was implemented in MATLAB 6.5. The first update of the step 6 in the algorithm involves a nonlinear constrained optimization problem. Although the optimization problem can be solved numerically, but we prefer, for our experiments, to keep the parameters $\theta^{s,*}$ fixed (i.e. $\theta^{s,*}|_{t+1} = \theta^{s,*}|_t$). The reason is that the computational complexity of the algorithm should be as minimum as possible for an analysis of the physiological data in real-time.

The algorithm 2 was initialized as follows:

1. The structure of the Takagi-Sugeno fuzzy filter that defines $K = 4$ number of rules (taking number of inputs $n = 2$ and defining 2 triangular membership functions for each of two inputs) was considered.
2. The matrix c and vector h were designed to incorporate the constrains on membership functions that any two consecutive knots must be separated at least by a distance of 0.1. Further, the extreme left knot must be greater than the minimum value of input variable in training set and the extreme right one less than the maximum value of input variable in the training set.
3. The prior m_0 was equal to the zero vector and Λ_0 was equal to the unity matrix.
4. The parameters $a_0 = 10^6, b_0 = 10^{-6}$ (i.e. relatively noninformative priors for the uncertainty) were chosen.
5. The algorithm 2 was started with $a|_0 = a_0$, $b|_0 = b_0$, and setting $\theta^*|_0$ for the uniformly distributed membership functions in the corresponding inputs' ranges.

For each subject, the two heartbeat interval models (first corresponding to the state of supine rest and the the second corresponding to the state of head-up tilt) were inferred. Let $(\theta^{rest}, \Lambda^{rest}, m^{rest}, a^{rest}, b^{rest})$ and $(\theta^{tilt}, \Lambda^{tilt}, m^{tilt}, a^{tilt}, b^{tilt})$ denote for a subject the model parameters (returned by algorithm 2) corresponding to the physiological states of rest and tilt respectively. These parameters act as priors for the

Table 1 The statistics of the features

feature	rest	tilt
F_1	182.9065 ± 34.0042	104.8991 ± 29.6306
F_2	138.3097 ± 37.9080	246.9050 ± 42.8579
F_3	0.8733 ± 0.1405	0.0358 ± 0.0887
F_4	0.1267 ± 0.1405	0.9642 ± 0.0887
F_5	188.8243 ± 31.6398	250 ± 37.9249
F_6	0.8774 ± 0.1420	0.0320 ± 0.0894
F_7	0.1226 ± 0.1420	0.9680 ± 0.0894

analysis algorithm 1. That is,

$$m_0^1 = m^{rest},\ \Lambda_0^1 = \Lambda^{rest},\ a_0^1 = a^{rest},\ b_0^1 = b^{rest},$$

$$m_0^2 = m^{tilt},\ \Lambda_0^2 = \Lambda^{tilt},\ a_0^2 = a^{tilt},\ b_0^2 = b^{tilt}.$$

Algorithm 1 additionally needs to choose the hyper-parameter c_0 which is taken equal to 2. The algorithm 1 was started with

$$q^*(s_i = s)|_0 = \frac{1}{S} = 0.5$$

$$m^s|_0 = m_0^s,\ \Lambda^s|_0 = \Lambda_0^s,\ b^s|_0 = b_0^s + \frac{\sum_{i=1}^N q^*(s_i = s)|_0}{2}$$

$$\frac{1}{a^s|_0} = \frac{1}{a_0^s} + \frac{1}{2}\Big[(Y^s_{q|_0} - B_{q|_0}(\theta^{s,*}|_0)m^s|_0)^T(Y^s_{q|_0} - B_{q|_0}(\theta^{s,*}|_0)m^s|_0) \\ +Tr\left((\Lambda^s|_0)^{-1}(B_{q|_0}(\theta^{s,*}|_0))^T B_{q|_0}(\theta^{s,*}|_0)\right)\Big]$$

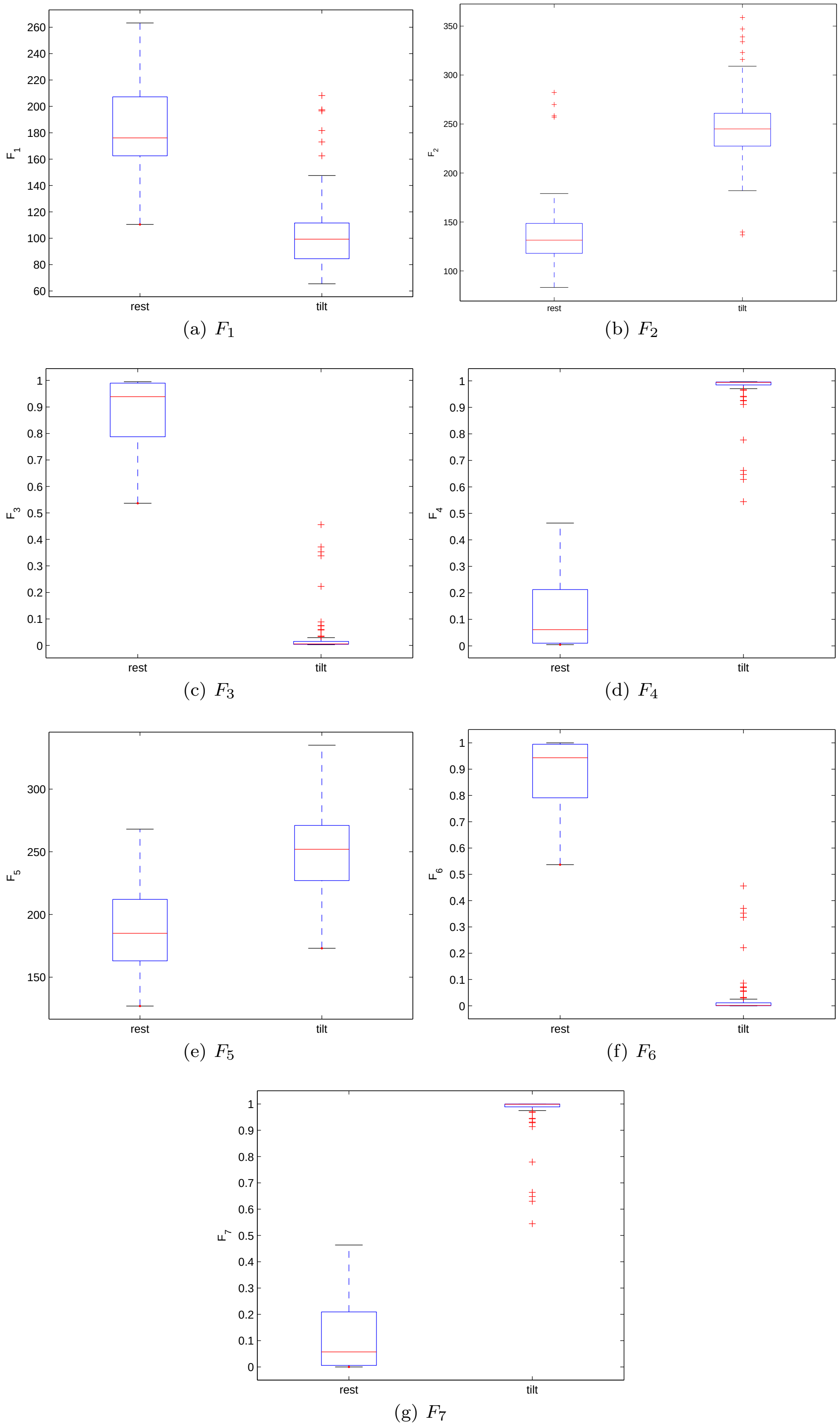
where $\theta^{1,*}|_0 = \theta^{rest}$ and $\theta^{2,*}|_0 = \theta^{tilt}$.

Algorithm 1 was run to analyze the 3 minutes long data segments. It is expected that the parameters returned by algorithm 1 should serve as the features which characterize the physiological state of the subject. The following features are defined based on the parameters returned by algorithm 1:

$$F_1 = b^1,\ F_2 = b^2,\ F_3 = d_1,\ F_4 = d_2,\ F_5 = c$$

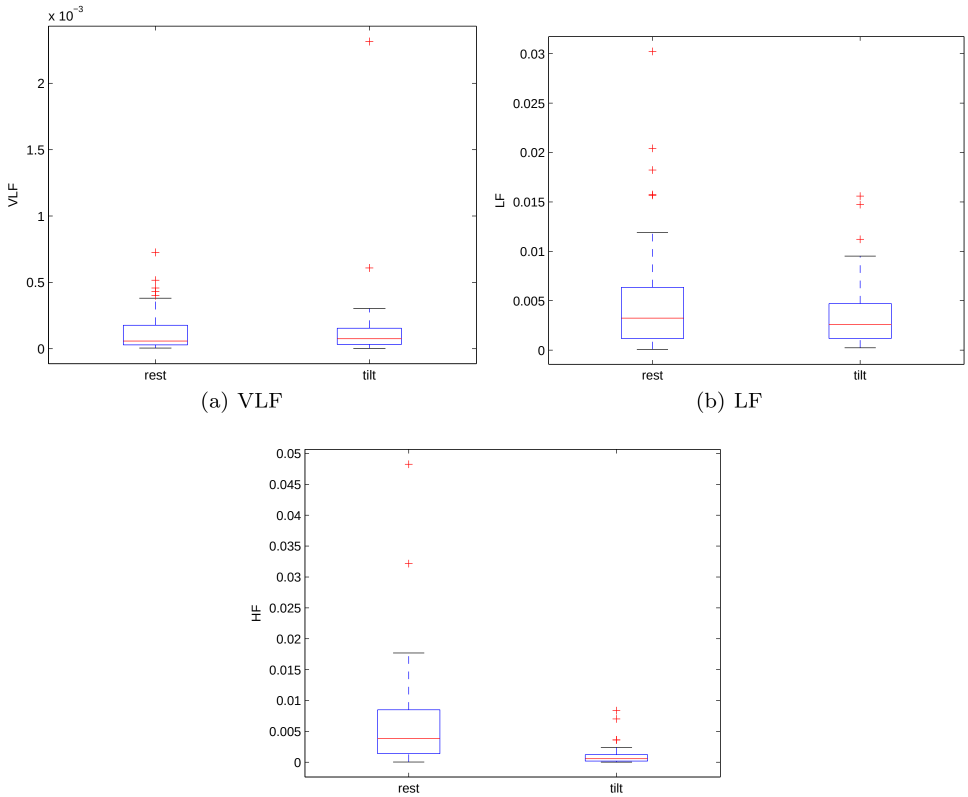
$$F_6 = \frac{\sum_{i=1}^N q^*(s_i = 1)}{N},\ F_7 = \frac{\sum_{i=1}^N q^*(s_i = 2)}{N}.$$

The statistics of the feature values is summarized in table 1 and figure 1. Table 1 reports for each physiological state the means and standard deviations of the extracted features. Figure 1 shows the box-and-whisker plots for the data. It is easy to observe from table 1 and figure 1 that variables $(F_1, F_2, \cdots, F_7)$ take significant different values for the state of supine rest from those for the state of head-up tilt. This verifies that the features $(F_1, F_2, \cdots, F_7)$ are potentially capable of distinguishing the two physiological states and hence are diagnostic efficient.



(a) F_1 (b) F_2 (c) F_3 (d) F_4 (e) F_5 (f) F_6 (g) F_7

Fig. 1 The box-and-whisker plots of features for supine rest and head-up tilt



(a) VLF (b) LF

(c) HF

Fig. 2 The box-and-whisker plots of HRV measures for supine rest and head-up tilt

The physiological state of a subject is predicted from the analysis of the data segment as follows:

$$\text{predicted state} = \begin{cases} \text{supine rest}, & \text{if } F_6 > F_7 \\ \text{head-up tilt}, & \text{if } F_7 > F_6 \end{cases}$$

The accuracy of 100% was observed in the physiological state prediction of the subjects.

The spectral analysis is a common method of extracting features of heartbeat intervals. To compare the proposed method with a standard method, the 3 minutes long inter-beat intervals segments were analyzed also by estimating the power spectral density (i.e. distribution of power as a function of frequency). Each series of inter-beat intervals was re-sampled at a constant rate of 4 Hz by using a cubic spline interpolation followed by a detrending [38]. Fast Fourier transform (FFT) was the method used for estimating the power spectral density. HRV spectrum is typically divided into three frequency bands: very low frequency VLF (0.01-0.04Hz), low frequency LF (0.04-0.15Hz), and high frequency HF (0.15-0.5Hz). The power components in VLF, LF, and HF frequency bands are the standard frequency domain measures of HRV [39]. Table 2 and figure 2 summarize the statistics of HRV measures for each physiological state. A comparison of the statistics of the two analysis methods demonstrates that the

Table 2 The statistics of the HRV measures

feature	rest	tilt
VLF	$1.2752e-004 \pm 1.4611e-004$	$1.2902e-004 \pm 2.7488e-004$
LF	0.0049 ± 0.0054	0.0034 ± 0.0030
HF	0.0060 ± 0.0073	$9.7146e-004 \pm 0.0013$

proposed features $(F_1, F_2, \cdots, F_7)$ are more efficient than some of the standard HRV indices (i.e. VLF, LF, HF power components) in distinguishing the physiological state of supine rest from that of head-up tilt.

7 Concluding Remarks

We have presented a new fuzzy filtering based stochastic modeling approach for an analysis of heartbeat intervals. VB framework was used for extracting the features of heartbeat interval series. The extracted features were observed to possess a remarkably high diagnostic efficiency and therefore the physiological states of the subjects were predicted with 100% accuracy in our experiments. The proposed modeling approach for the analysis of physiological signals may be helpful in solving various medical decision making problems. The method took simultaneously the advantages of probability theory and fuzzy filtering. The proposed approach achieves a high diagnostic efficiency as a result of mathematically formulating the problem in a sensible way. However, there remain some limitations of the method which will be overcome in our future research. These limitations include

- The physiological meaning of the features extracted from heartbeat intervals is not clear. These feature are related to the probabilities of occurrence of a physiological state in the data.
- The first update of the step 6 in the proposed algorithm involves a nonlinear constrained optimization problem. However, a closed-form expression for the updating of nonlinear membership functions related parameters, like other parameters updates, should be developed. This is computationally the most expensive step of the algorithm and therefore, as stated in remark 3, the membership functions related parameters were kept fixed in view of a real-time operation.

A Preliminaries from [17]

A.1 A Takagi-Sugeno Fuzzy Filter

Consider a Takagi-Sugeno fuzzy model $(F_s : X \to Y)$ that maps $n-$dimensional real input space $(X = X_1 \times X_2 \times \cdots \times X_n)$ to one dimensional real line.

A.1.1 Grid Partitioning of Input Space

A rule of the model is represented as

$$\text{If } x_1 \text{ is } A_1 \text{ and } \cdots \text{ and } x_n \text{ is } A_n \text{ then } y_f = s_0 + s_1 x_1 + \cdots + s_n x_n.$$

Here $(x_1, \cdots, x_n)$ are the model input variables, y_f is the filtered output variable, $(A_1, \cdots, A_n)$ are the linguistic terms which are represented by fuzzy sets, and $(s_0, s_1, \cdots, s_n)$ are real scalars. Given a universe of discourse X_j, a fuzzy subset A_j of X_j is characterized by a mapping:

$$\mu_{A_j} : X_j \to [0, 1],$$

where for $x_j \in X_j$, $\mu_{A_j}(x_j)$ can be interpreted as the degree or grade to which x_j belongs to A_j. This mapping is called as membership function of the fuzzy set. Let us define, for j^{th} input, P_j non-empty fuzzy subsets of X_j (represented by $A_{1j}, A_{2j}, \cdots, A_{P_j j}$). Let the i^{th} rule of the rule-base is represented as

$$R_i : \text{If } x_1 \text{ is } A_{i1} \text{ and } \cdots \text{ and } x_n \text{ is } A_{in} \text{ then } y_f = s_{i0} + s_{i1}x_1 + \cdots + s_{in}x_n,$$

where $A_{i1} \in \{A_{11}, \cdots, A_{P_1 1}\}, A_{i2} \in \{A_{12}, \cdots, A_{P_2 2}\}$ and so on. Now, the different choices of $A_{i1}, A_{i2}, \cdots, A_{in}$ leads to the $K = \prod_{j=1}^{n} P_j$ number of fuzzy rules. For a given input vector $x = [x_1 \cdots x_n]^T \in R^n$, the *degree of fulfillment* of the i^{th} rule, by modeling the logic operator 'and' using product, is given by

$$g_i(x) = \prod_{j=1}^{n} \mu_{A_{ij}}(x_j).$$

The output of the fuzzy model to input vector x is computed by taking the weighted average of the output provided by each rule:

$$y_f = \frac{\sum_{i=1}^{K} (s_{i0} + s_{i1}x_1 + \cdots + s_{in}x_n) g_i(x)}{\sum_{i=1}^{K} g_i(x)} = \frac{\sum_{i=1}^{K} (s_{i0} + s_{i1}x_1 + \cdots + s_{in}x_n) \prod_{j=1}^{n} \mu_{A_{ij}}(x_j)}{\sum_{i=1}^{K} \prod_{j=1}^{n} \mu_{A_{ij}}(x_j)}. \tag{5}$$

Let us define a real vector θ such that the membership functions of any type (e.g. trapezoidal, triangular, etc) can be constructed from the elements of vector θ. To illustrate the construction of membership functions based on knot vector (θ), consider the following examples:

Triangular membership functions Let

$$\theta = (t_1^0, t_1^1, \cdots, t_1^{P_1-2}, t_1^{P_1-1}, \cdots, t_n^0, t_n^1, \cdots, t_n^{P_n-2}, t_n^{P_n-1})$$

such that for i^{th} input, $t_i^0 < t_i^1 < \cdots < t_i^{P_i-2} < t_i^{P_i-1}$ holds for all $i = 1, \cdots, n$. Now, P_i triangular membership functions for i^{th} input $(\mu_{A_{1i}}, \mu_{A_{2i}}, \cdots, \mu_{A_{P_i i}})$ can be defined as:

$$\begin{aligned}
\mu_{A_{1i}}(x_i, \theta) &= \max\left(0, \min\left(1, \frac{t_i^1 - x_i}{t_i^1 - t_i^0}\right)\right) \\
\mu_{A_{ji}}(x_i, \theta) &= \max\left(0, \min\left(\frac{x_i - t_i^{j-2}}{t_i^{j-1} - t_i^{j-2}}, \frac{t_i^j - x_i}{t_i^j - t_i^{j-1}}\right)\right), \ j = 2, \cdots, P_i - 1 \\
\mu_{A_{P_i i}}(x_i, \theta) &= \max\left(0, \min\left(\frac{x_i - t_i^{P_i-2}}{t_i^{P_i-1} - t_i^{P_i-2}}, 1\right)\right)
\end{aligned}$$

One-dimensional clustering criterion based membership functions Let

$$\theta = (t_1^0, t_1^1, \cdots, t_1^{P_1-2}, t_1^{P_1-1}, \cdots, t_n^0, t_n^1, \cdots, t_n^{P_n-2}, t_n^{P_n-1})$$

such that for i^{th} input, $t_i^0 < t_i^1 < \cdots < t_i^{P_i-2} < t_i^{P_i-1}$ holds for all $i = 1, \cdots, n$. Consider the problem of assigning two different memberships (say $\mu_{A_{1i}}$ and $\mu_{A_{2i}}$) to a point x_i such that $t_i^0 < x_i < t_i^1$, based on following clustering criterion:

$$[\mu_{A_{1i}}(x_i), \mu_{A_{2i}}(x_i)] = \arg \min_{[u_1, u_2]} \left[u_1^2(x_i - t_i^0)^2 + u_2^2(x_i - t_i^1)^2, \ u_1 + u_2 = 1\right].$$

This results in

$$\mu_{A_{1i}}(x_i) = \frac{(x_i - t_i^1)^2}{(x_i - t_i^0)^2 + (x_i - t_i^1)^2},\ \mu_{A_{2i}}(x_i) = \frac{(x_i - t_i^0)^2}{(x_i - t_i^0)^2 + (x_i - t_i^1)^2}.$$

Thus, for i^{th} input, P_i membership functions can be defined as:

$$\mu_{A_{1i}}(x_i,\theta) = \begin{cases} 1 & x_i \le t_i^0 \\ \dfrac{(x_i - t_i^1)^2}{(x_i - t_i^0)^2 + (x_i - t_i^1)^2} & t_i^0 \le x_i \le t_i^1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{A_{ji}}(x_i,\theta) = \begin{cases} \dfrac{(x_i - t_i^{j-2})^2}{(x_i - t_i^{j-2})^2 + (x_i - t_i^{j-1})^2} & t_i^{j-2} \le x_i \le t_i^{j-1} \\ \dfrac{(x_j - t_i^{j})^2}{(x_i - t_i^{j-1})^2 + (x_i - t_i^{j})^2} & t_i^{j-1} \le x_i \le t_i^{j} \\ 0 & \text{otherwise} \end{cases},\ j = 2, \cdots, P_i - 1$$

$$\mu_{A_{P_i i}}(x_i,\theta) = \begin{cases} 1 & x_i \ge t_i^{P_i-1} \\ \dfrac{(x_i - t_i^{P_i-2})^2}{(x_i - t_i^{P_i-2})^2 + (x_i - t_i^{P_i-1})^2} & t_i^{P_i-2} \le x_i \le t_i^{P_i-1} \\ 0 & \text{otherwise} \end{cases}$$

Gaussian membership functions Let

$$\theta = (t_1^0, t_1^1, \cdots, t_1^{P_1-2}, t_1^{P_1-1}, \cdots, t_n^0, t_n^1, \cdots, t_n^{P_n-2}, t_n^{P_n-1})$$

such that for i^{th} input, $t_i^0 < t_i^1 < \cdots < t_i^{P_i-2} < t_i^{P_i-1}$ holds for all $i = 1, \cdots, n$. Now, P_i Gaussian membership functions for i^{th} input can be defined as:

$$\mu_{A_{ji}}(x_i,\theta) = e^{-(x_i - t_i^{j-1})^2},\ j = 1, \cdots, P_i$$

For any choice of membership functions (which can be constructed from a vector θ), (5) can be rewritten as function of θ:

$$y_f = \sum_{i=1}^{K} (s_{i0} + s_{i1}x_1 + \cdots + s_{in}x_n)\tilde{G}_i(x,\theta),\ \tilde{G}_i(x,\theta) = \frac{\prod_{j=1}^{n} \mu_{A_{ij}}(x_j,\theta)}{\sum_{i=1}^{K}\prod_{j=1}^{n} \mu_{A_{ij}}(x_j,\theta)}.$$

Let us introduce the following notation:

$$\alpha = [\, s_{10}\ s_{11}\ \cdots\ s_{1n}\ \cdots\ s_{K0}\ s_{K1}\ \cdots\ s_{Kn}\,]^T \in R^{K(n+1)}$$

$$G(x,\theta) = [\, \tilde{G}_1(x,\theta)\ x^T\tilde{G}_1(x,\theta)\ \cdots\ \tilde{G}_K(x,\theta)\ x^T\tilde{G}_K(x,\theta)\,]^T \in R^{K(n+1)}$$

Now, we have

$$y_f = G^T(x,\theta)\alpha.$$

In this expression, θ is not allowed to be any arbitrary vector, since the elements of θ must $\forall i = 1, \cdots, n$, ensure

$$a_i \le t_i^0 < t_i^1 < \cdots < t_i^{P_i-2} < t_i^{P_i-1} \le b_i,\ \text{where } x_i \in [a_i, b_i].$$

These inequalities and any other membership functions related constraints (designed for incorporating a priori knowledge) can be written in the form of a matrix inequality $c\theta \ge h$. Hence, a Takagi-Sugeno type fuzzy filter can be represented as

$$y_f = G^T(x,\theta)\alpha,\ c\theta \ge h.$$

A.1.2 Fuzzy Clustering Based Partitioning of Input Space

Several studies have used fuzzy c-mean (or its robust alternatives) to find clusters in the input space and thus obtaining the parameters of the membership functions. Such methods define multivariate membership functions and corresponding to each cluster, there exists a fuzzy rule of the Takagi-Sugeno form:

$$R_i : \text{If } x \text{ is } A_i \text{ then } y_f = s_{i0} + s_{i1}x_1 + \cdots + s_{in}x_n,\ i = 1, 2, \cdots, K$$

The fuzzy set A_i (with a membership function $A_i(x) : R^n \rightarrow [0, 1]$) is typically defined with the Gaussian function:

$$\mu_{A_i}(x) = \prod_{j=1}^{n} \exp(-\frac{|x_j - t_j^{i,0}|^2}{2t_j^{i,1}})$$

where $t_j^{i,0}$ is the center and $t_j^{i,1}$ is the dispersion of the membership function on x_j defined by the $i-$th cluster. The parameters of the membership functions (i.e. $t_j^{i,0}, t_j^{i,1};\ j = 1, \cdots, n;\ i = 1, \cdots, K$) can be obtained from fuzzy clustering of the input data, see e.g. [27]. Let the parameters of the membership functions are collected in a vector θ, defined as

$$\theta = (t_1^{1,0}, t_1^{1,1}, \cdots, t_n^{1,0}, t_n^{1,1}, \cdots, t_1^{K,0}, t_1^{K,1}, \cdots, t_n^{K,0}, t_n^{K,1}).$$

It is easy now to see that the fuzzy filter, like the grid partitioning case, can be still functionally represented as

$$y_f = G^T(x, \theta)\alpha,\ \text{where}$$

$$\alpha = [\, s_{10}\ s_{11}\ \cdots\ s_{1n}\ \cdots\ s_{K0}\ s_{K1}\ \cdots\ s_{Kn}\,]^T \in R^{K(n+1)}$$

$$G(x, \theta) = [\, \tilde{G}_1(x, \theta)\ x^T\tilde{G}_1(x, \theta)\ \cdots\ \tilde{G}_K(x, \theta)\ x^T\tilde{G}_K(x, \theta)\,]^T \in R^{K(n+1)}$$

$$\tilde{G}_i(x, \theta) = \frac{\mu_{A_i}(x, \theta)}{\sum_{i=1}^{K} \mu_{A_i}(x, \theta)}.$$

Although the parameters of membership functions (i.e. elements of vector θ) are obtained by fuzzy clustering, one may prefer a fine tuning of the elements of vector θ under a filtering performance criterion. In this case, the tuning process can be constrained. A necessary constraint is that the variances of Gaussian membership functions must be greater than zero. Any type of constraints on the parameters of membership functions can be formulated as a matrix inequality $c\theta \geq h$.

A.2 An algorithm for VB inference of fuzzy filter parameters

Algorithm 2 An algorithm for VB inference of fuzzy filter parameters from [17]

Require: Data pairs $\{x(j), y(j)\}_{j=1,\cdots,N}$.

1: Choose a fuzzy filter structure $\mathfrak{m}$; hyper-parameters m_0, Λ_0, a_0, b_0 which define the regularizing priors; parameters c, h such that the interpretability constraints on the membership functions can be formulated as $c\theta \geq h$.

2: Set iteration count $t = 0$ and choose a tolerance limit (say, equal to 0.01%).

3: **if** $(\mathcal{F}|_{t+1} - \mathcal{F}|_t < 0.0001\mathcal{F}|_t)$ & $(t > 0)$ **then**

4: **return** $\theta^*|_{t+1}$, $\Lambda|_{t+1}$, $m|_{t+1}$, $a|_{t+1}$, $b|_{t+1}$.

5: **else**

6: Update the parameters as follows

$$\theta^*|_{t+1} = \arg\min_{\theta} \left[r(m|_t, \Lambda|_t, \theta);\ c\theta \geq h \right].$$

$$\Lambda|_{t+1} = \Lambda_0 + a|_t b|_t (B(\theta^*|_{t+1}))^T B(\theta^*|_{t+1})$$

$$m|_{t+1} = \left[\Lambda_0 + a|_t b|_t (B(\theta^*|_{t+1}))^T B(\theta^*|_{t+1})\right]^{-1} \left[\Lambda_0 m_0 + a|_t b|_t (B(\theta^*|_{t+1}))^T Y\right]$$

$$b|_{t+1} = b_0 + \frac{N}{2}$$

$$\frac{1}{a|_{t+1}} = \frac{1}{a_0} + \frac{1}{2} r(m|_{t+1}, \Lambda|_{t+1}, \theta^*|_{t+1})$$

Here, $(m|_0, \Lambda|_0, a|_0, b|_0, \theta^*|_0)$ denote an initial guess,

$$r(m, \Lambda, \theta) = (Y - B(\theta)m)^T (Y - B(\theta)m) + Tr\left((\Lambda)^{-1} (B(\theta))^T B(\theta)\right)$$

and $\mathcal{F}|_t$ is computed as

$$\begin{aligned}\mathcal{F}|_t = {} & \frac{N}{2}(\Psi(b|_t) + \log(a|_t)) - \frac{N}{2}\log(2\text{pi}) - \frac{a|_t b|_t}{2} r(m|_t, \Lambda|_t, \theta^*|_t) - \frac{1}{2}\log\left(\frac{|(\Lambda_0)^{-1}|}{|(\Lambda|_t)^{-1}|}\right) \\ & -\frac{1}{2} Tr\left(\Lambda_0 (\Lambda|_t)^{-1}\right) - \frac{1}{2}(m|_t - m_0)^T \Lambda_0 (m|_t - m_0) + \frac{K}{2} - \log(\Gamma(b_0)) \\ & -b_0 \log(a_0) + \log(\Gamma(b|_t)) + b_0 \log(a|_t) - b|_t \Psi(b|_t) + b_0 \Psi(b|_t) - b|_t \frac{a|_t}{a_0} + b|_t.\end{aligned}$$

7: **end if**

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